A GENERALIZATION OF THE PICARD-BRAUER EXACT SEQUENCE

CRISTIAN D. GONZÁLEZ-AVILÉS

Abstract. We extend an argument of S.Lichtenbaum involving codimension one cycles to higher codimensions and obtain a generalization of the well-known Picard-Brauer exact sequence for a smooth variety X. The resulting exact sequence connects the codimension n Chow group of X with a certain "Brauer-like" group.

1. Introduction.

Let k be a field and let X be a geometrically integral algebraic k-scheme. We write $\overline{k}$ for a fixed separable algebraic closure of k and set $\varGamma = \mathrm{Gal}\big(\overline{k}/k\big)$. The $\overline{k}$-scheme $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\,\overline{k}$ will be denoted by $\overline{X}$. Let $\overline{k}[X]^* = H^0_{\text{ét}}(\overline{X}, \mathbb{G}_m)$ and $\mathrm{Br}'X = H^2_{\text{ét}}(X, \mathbb{G}_m)$ be, respectively, the group of invertible regular functions on $\overline{X}$ and the cohomological Brauer group of X. The exact sequence mentioned in the title is the familiar exact sequence

(1)
$$\begin{aligned} 0 \;&\to\; H^1(k, \overline{k}[X]^*) \to \mathrm{Pic}\, X \to \big(\mathrm{Pic}\,\overline{X}\big)^{\varGamma} \to H^2(k, \overline{k}[X]^*) \to \mathrm{Br}'_1 X \\ &\to\; H^1\big(k, \mathrm{Pic}\,\overline{X}\big) \to H^3(k, \overline{k}[X]^*) \end{aligned}$$

where $H^i(k, -) = H^i(\varGamma, -)$ and $\mathrm{Br}'_1 X = \mathrm{Ker}\left(\mathrm{Br}'X \to \mathrm{Br}'\,\overline{X}\right)$. This sequence may be obtained from the exact sequence of terms of low degree belonging to the Hochschild-Serre spectral sequence

$$H^r\big(k, H^s_{\text{ét}}\big(\overline{X}, \mathbb{G}_m\big)\big) \implies H^{r+s}_{\text{ét}}(X, \mathbb{G}_m).$$

When X is *smooth* (which we assume from now on), there exists an alternative derivation of (1) which makes use of the following (no less familiar) exact sequence:

$$0 \to \overline{k}[X]^* \to \overline{k}(X)^* \to \mathrm{Div}\,\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0 \tag{2}$$

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where $\overline{k}(X)^*$ (resp. $\mathrm{Div}\overline{X}$) is the group of invertible rational functions (resp. Cartier divisors) on $\overline{X}$. This approach, seemingly first used by S.Lichtenbaum in [4] and then reconsidered by Yu.Manin [5, p.403], consists in splitting (2) into two short exact sequences of Γ-modules and then taking Γ-cohomology of these sequences. The resulting long Γ-cohomology sequences are then appropriately combined to produce (1). This paper is a generalization of this idea. The key observation to make is that (2) may be seen as arising from the Gersten-Quillen complex corresponding to the Zariski sheaf $\mathcal{K}_{1,\overline{X}}$, which is the sheaf on $\overline{X}$ associated to the presheaf $U \mapsto K_1(U) = H^0(U, \mathcal{O}_U)^*$. In Section 2 we work with the Gersten-Quillen complex corresponding to the Zariski sheaf $\mathcal{K}_{n,\overline{X}}$ associated to the presheaf $U \mapsto K_n(U)$, where K_n is Quillen's n-th K-functor ($1 \leq n \leq d = \dim(X)$), and obtain the following result. Let $\partial^{n-1}\colon \bigoplus_{y\in\overline{X}^{n-1}} \overline{k}(y)^* \to Z^n\big(\overline{X}\big)$ be the "sum of divisors" map and let $B_n(X)$ be the kernel of the induced map

$$H^2\Big(k, \bigoplus_{y\in\overline{X}^{n-1}} \overline{k}(y)^*\Big) \to H^2\big(k, Z^n\big(\overline{X}\big)\big).$$

Main Theorem. *Let X be a smooth, geometrically integral, algebraic k-scheme. Then there exists a natural exact sequence*

$$\begin{aligned} 0 \;\to\; & H^1(k, \mathrm{Ker}\,\partial^{n-1}) \to CH^n(X) \to CH^n\big(\overline{X}\big)^{\Gamma} \to H^2(k, \mathrm{Ker}\,\partial^{n-1}) \\ \to\; & B_n(X) \to H^1\big(k, CH^n\big(\overline{X}\big)\big) \to H^3(k, \mathrm{Ker}\,\partial^{n-1}). \end{aligned}$$

The case $n = 1$ of the theorem is precisely the exact sequence (1).

In Section 4, which concludes the paper, we show that the group $B_n(X)$ in the exact sequence of the theorem is "Brauer-like", in the sense that it contains a copy of $\mathrm{Br}_1 Y = \mathrm{Ker}\big[\mathrm{Br}\,Y \to \mathrm{Br}\,\overline{Y}\,\big]$ for every smooth closed integral subscheme $Y \subset X$ of codimension $n-1$.

2. Preliminaries

We keep the notations of the Introduction. In particular, X is a smooth, geometrically integral algebraic k-scheme of dimension d and n denotes a fixed integer such that $1 \leq n \leq d$.

There exists a natural bijection between the set of schematic points of X and the set of closed integral subschemes of X. This is defined by associating to a point $x \in X$ the schematic closure $V(x)$ of x in X. The codimension (resp. dimension) of x is by definition the codimension (resp. dimension) of $V(x)$. The set of points of X of codimension (resp. dimension) i will be denoted by X^i (resp. X_i), and η (resp. $\overline{\eta}$) will denote the generic point of X (resp. $\overline{X}$). If $x \neq \eta$, the function field of $V(x)$ will be denoted by $k(x)$. We use the standard notation $k(X)$

for the function field of $X = V(\eta)$. For each $x \in X$, i_x will denote the canonical map $\operatorname{Spec} k(x) \to X$. The function field of $\overline{X}$ will be denoted by $\overline{k}(X)$. For simplicity, we will write $V(\overline{x})$ for $V(x) \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$.

Since $\overline{X}$ is regular [3, 6.7.4], the sheaf $\mathcal{K}_{n,\overline{X}}$ admits the following flasque resolution, known as the Gersten-Quillen resolution (see [7, p.72]):

$$\begin{aligned} 0 \to \mathcal{K}_{n,\overline{X}} \ &\to (i_{\overline{\eta}})_* K_n \overline{k}(X) \to \bigoplus_{y \in \overline{X}^1} (i_y)_* K_{n-1} \overline{k}(y) \to \dots \\ &\to \bigoplus_{y \in \overline{X}^{n-1}} (i_y)_* \overline{k}(y)^* \to \bigoplus_{y \in \overline{X}^{n}} (i_y)_* \mathbb{Z} \ \to 0 \end{aligned}$$

where, for $y \in \overline{X}^i$, $K_{n-i}\,\overline{k}(y)$ is regarded as a constant sheaf on $\overline{k}(y)$. It follows that the groups $H^i\big(\overline{X}, \mathcal{K}_{n,\overline{X}}\big) = H^i\big(\overline{X}, \mathcal{K}_n\big)$ are the cohomology groups of the complex

$$(3)\quad K_n \overline{k}(X) \xrightarrow{\partial^0} \bigoplus_{y \in \overline{X}^1} K_{n-1} \overline{k}(y) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-2}} \bigoplus_{y \in \overline{X}^{n-1}} \overline{k}(y)^* \xrightarrow{\partial^{n-1}} \bigoplus_{y \in \overline{X}^{n}} \mathbb{Z}\,.$$

Now, if $q\colon \overline{X} \to X$ is the canonical morphism and $x \in X$, we write $\overline{X}_x^{n-i}$ for the set of points $y \in \overline{X}^{n-i}$ such that $q(y) = x$. For $i = 1, 2, \dots, n-1$ and $x \in X^{n-i}$, set

$$\overline{K}_i(x) = \bigoplus_{y \in \overline{X}_x^{n-i}} K_i \overline{k}(y).$$

Further, write $Z^n\big(\overline{X}\big)$ for the group of codimension n cycles on $\overline{X}$, i.e.,

$$Z^n\big(\overline{X}\big) = \bigoplus_{y \in \overline{X}^n} \mathbb{Z}\,.$$

Then (3) may be written as

$$(4)\quad K_n \overline{k}(X) \xrightarrow{\partial^0} \bigoplus_{x \in X^1} \overline{K}_{n-1}(x) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-2}} \bigoplus_{x \in X^{n-1}} \overline{K}_1(x) \xrightarrow{\partial^{n-1}} Z^n\big(\overline{X}\big)\,.$$

The differential ∂^{n-1} equals $\sum_{x \in X^{n-1}} \partial_x^{n-1}$, where, for each $x \in X^{n-1}$,

$$\partial_x^{n-1}\colon \overline{K}_1(x) = \bigoplus_{y \in \overline{X}_x^{n-1}} \overline{k}(y)^* \to Z^n\big(\overline{X}\big)$$

is the sum of the divisor maps

$$\operatorname{div}_y\colon \overline{k}(y)^* \to Z^n\big(\overline{X}\big).$$

For definition of the latter, see [7, p.72]. We note that each of the maps div_y factors through $Z^1(V(y))$, whence each ∂_x^{n-1} factors through $Z^1(V(\overline{x}))$.

We will write $CH^n(X)$ for the Chow group of codimension n cycles on X modulo rational equivalence. Then $H^n(X,\mathcal{K}_n)=CH^n(X)$ ("Bloch's formula").

3. Proof of the main theorem

The complex (4) induces the following short exact sequences of Γ-modules:

$$0\to \mathrm{Im}\,\partial^{n-1}\to Z^n\big(\overline{X}\big)\to CH^n\big(\overline{X}\big)\to 0 \tag{5}$$

and

$$0\to \mathrm{Ker}\,\partial^{n-1}\to \bigoplus_{x\in X^{n-1}}\overline{K}_1(x)\to \mathrm{Im}\,\partial^{n-1}\to 0. \tag{6}$$

Observe that the natural morphism $q\colon \overline{X}\to X$ induces a homomorphism $CH^n(X)\to CH^n\big(\overline{X}\big)^{\Gamma}$.

Lemma 3.1. *There exist canonical isomorphisms*

$$\mathrm{Ker}\Big[CH^n(X)\to CH^n\big(\overline{X}\big)^{\Gamma}\Big] \;=\; H^1\big(k,\mathrm{Ker}\,\partial^{n-1}\big)$$

$$\mathrm{Coker}\Big[CH^n(X)\to CH^n\big(\overline{X}\big)^{\Gamma}\Big] \;=\; H^1\big(k,\mathrm{Im}\,\partial^{n-1}\big)$$

and a canonical exact sequence

$$0\to H^1\big(k,CH^n\big(\overline{X}\big)\big)\to H^2\big(k,\mathrm{Im}\,\partial^{n-1}\big)\to H^2\big(k,Z^n\big(\overline{X}\big)\big).$$

Proof. This follows by taking Γ-cohomology of (5), using the fact that $Z^n(\overline{X})$ is a permutation Γ-module and arguing as in [1, proof of Proposition 3.6] to establish the first isomorphism. □

Lemma 3.2. *The exact sequence (6) induces an exact sequence*

$$\begin{aligned}0 \;\to\;& H^1\big(k,\mathrm{Im}\,\partial^{n-1}\big)\to H^2\big(k,\mathrm{Ker}\,\partial^{n-1}\big)\to \bigoplus_{x\in X^{n-1}} H^2\big(k,\overline{K}_1(x)\big)\\ \to\;& H^2\big(k,\mathrm{Im}\,\partial^{n-1}\big)\to H^3\big(k,\mathrm{Ker}\,\partial^{n-1}\big)\,.\end{aligned}$$

Proof. By Shapiro's Lemma, for each $x\in X^{n-1}$ there exists a (non-canonical) isomorphism

$$H^*(k,\overline{K}_1(x))\simeq H^*\big(\mathrm{Gal}\big(\overline{k}(y)/k(x)\big),\overline{k}(y)^*\big)$$

where, on the right, we have chosen a point $y\in \overline{X}^{\,n-1}$ such that $q(y)=x$. The result now follows by taking Γ-cohomology of (6), using Hilbert's Theorem 90. □

Combining Lemmas 3.1 and 3.2, we obtain

Proposition 3.3. *There exists a canonical exact sequence*

$$\begin{aligned} 0 \;\to\; & H^1\big(k, \operatorname{Ker}\partial^{n-1}\big) \to CH^n(X) \to CH^n\big(\overline{X}\big)^{\Gamma} \\ \to\; & H^2\big(k, \operatorname{Ker}\partial^{n-1}\big) \to \bigoplus_{x\in X^{n-1}} H^2(k, \overline{K}_1(x)). \quad \square \end{aligned}$$

Now define

$$B_n(X) = \operatorname{Ker}\left[H^2\big(k, \textstyle\bigoplus_{x\in X^{n-1}} \overline{K}_1(x)\big) \to H^2\big(k, Z^n\big(\overline{X}\big)\big)\right], \tag{7}$$

where the map involved is induced by ∂^{n-1}. Since the composite

$$\operatorname{Ker}\partial^{n-1} \to \bigoplus_{x\in X^{n-1}} \overline{K}_1(x) \xrightarrow{\partial^{n-1}} Z^n\big(\overline{X}\big)$$

is zero, the natural map $H^2(k, \operatorname{Ker}\partial^{n-1}) \to \bigoplus_{x\in X^{n-1}} H^2(k, \overline{K}_1(x))$ factors through $B_n(X)$. Thus Proposition 3.3 yields a natural exact sequence

$$\begin{aligned} 0 \;\to\; & H^1\big(k, \operatorname{Ker}\partial^{n-1}\big) \to CH^n(X) \to CH^n\big(\overline{X}\big)^{\Gamma} \\ \to\; & H^2\big(k, \operatorname{Ker}\partial^{n-1}\big) \to B_n(X). \end{aligned} \tag{8}$$

We will now extend the above exact sequence by defining a map $B_n(X) \to H^1(k, CH^n\big(\overline{X}\big))$ whose kernel is exactly the image of the map $H^2(k, \operatorname{Ker}\partial^{n-1}) \to B_n(X)$ appearing in (8).

It is not difficult to check that the map

$$\bigoplus_{x\in X^{n-1}} H^2\big(k, \overline{K}_1(x)\big) \to H^2\big(k, \operatorname{Im}\partial^{n-1}\big)$$

intervening in the exact sequence of Lemma 3.2 maps $B_n(X)$ into the kernel of the map $H^2(k, \operatorname{Im}\partial^{n-1}) \to H^2\big(k, Z^n\big(\overline{X}\big)\big)$. The latter is naturally isomorphic to $H^1\big(k, CH^n\big(\overline{X}\big)\big)$ (see Lemma 3.1). Thus there exists a canonical map $B_n(X) \to H^1\big(k, CH^n\big(\overline{X}\big)\big)$. Again, it is not difficult to check that the kernel of the map just defined is exactly the image of the map $H^2(k, \operatorname{Ker}\partial^{n-1}) \to B_n(X)$ appearing in (8). Thus we obtain a natural exact sequence

$$\begin{aligned} 0 \;\to\; & H^1(k, \operatorname{Ker}\partial^{n-1}) \to CH^n(X) \to CH^n\big(\overline{X}\big)^{\Gamma} \to H^2(k, \operatorname{Ker}\partial^{n-1}) \\ \to\; & B_n(X) \to H^1(k, CH^n\big(\overline{X}\big)). \end{aligned}$$

Finally, the homomorphisms $H^1(k, CH^n\big(\overline{X}\big)) \to H^2(k, \operatorname{Im}\partial^{n-1})$ and $H^2(k, \operatorname{Im}\partial^{n-1}) \to H^3(k, \operatorname{Ker}\partial^{n-1})$ from Lemmas 3.1 and 3.2 induce a map $H^1(k, CH^n\big(\overline{X}\big)) \to H^3(k, \operatorname{Ker}\partial^{n-1})$ whose kernel is exactly the image of the map $B_n(X) \to H^1(k, CH^n\big(\overline{X}\big))$ defined above. Thus the following holds.

Theorem 3.4. *Let X be a smooth k-variety. Then there exists a natural exact sequence*

$$\begin{aligned} 0 \;\to\; & H^1(k, \operatorname{Ker}\partial^{\,n-1}) \to CH^n(X) \to CH^n\big(\overline{X}\big)^{\Gamma} \to H^2(k, \operatorname{Ker}\partial^{\,n-1}) \\ \to\; & B_n(X) \to H^1\big(k, CH^n\big(\overline{X}\big)\big) \to H^3(k, \operatorname{Ker}\partial^{\,n-1}), \end{aligned}$$

where $B_n(X)$ is the group (7).

Remark 3.5. When $n = 1$, there are natural isomorphisms $CH^n(X) = \operatorname{Pic} X$ and $CH^n\big(\overline{X}\big) = \operatorname{Pic}\overline{X}$ [3, 21.6.10 and 21.11.1]. Further, $X^{n-1} = \{\eta\}$, $\partial^{\,n-1} = \partial^{\,n-1}_{\eta} \colon \overline{K}_1(\eta) = \overline{k}(X)^* \to \operatorname{Div}\overline{X}$ is the usual divisor map (whose kernel equals $H^0(\overline{X}, \mathbb{G}_m) \overset{\text{def.}}{=} \overline{k}[X]^*$) and

$$B_n(X) = B_1(X) = \operatorname{Ker}\big[H^2(k, \overline{k}(X)^*) \to H^2\big(k, \operatorname{Div}\overline{X}\,\big)\big] = \operatorname{Br}_1 X,$$

where $\operatorname{Br}_1 X = \operatorname{Ker}\big(\operatorname{Br} X \to \operatorname{Br}\overline{X}\,\big)$ (see the next section). Thus the exact sequence of the theorem is indeed a generalization of (1).

4. The group $B_n(X)$

In this Section we show that the group $B_n(X)$ appearing in the exact sequence of Theorem 3.4 contains a copy of $\operatorname{Br}_1 Y = \operatorname{Ker}\big(\operatorname{Br} Y \to \operatorname{Br}\overline{Y}\,\big)$ for *every* smooth closed integral subscheme $Y \subset X$ of codimension $n-1$.

Recall that $\partial^{\,n-1} = \sum_{x \in X^{n-1}} \partial^{\,n-1}_x$, where, for each $x \in X^{n-1}$,

$$\partial^{\,n-1}_x \colon \overline{K}_1(x) = \bigoplus_{y \in \overline{X}^{\,n-1}_x} \overline{k}(y)^* \to Z^1\big(V(\overline{x})\big)$$

is the sum of divisors map. For each $x \in X^{n-1}$, set

$$B_n(x) = \operatorname{Ker}\big[\, H^2(k, \overline{K}_1(x)) \to H^2(k, Z^1(V(\overline{x})))\big]\,,$$

where the map involved is induced by $\partial^{\,n-1}_x$, and let

$$\Sigma \,:\, \bigoplus_{x \in X^{n-1}} H^2(k, Z^1(V(\overline{x}))) \to H^2(k, Z^n(\overline{X}))$$

be the natural map $(\xi_x) \mapsto \sum c_x(\xi_x)$, where $c_x \colon H^2(k, Z^1(V(\overline{x}))) \to H^2(k, Z^n(\overline{X}))$ is induced by the inclusion $Z^1(V(\overline{x})) \subset Z^n(\overline{X})$. Then there exists a canonical exact sequence

$$0 \to \bigoplus_{x \in X^{n-1}} B_n(x) \to B_n(X) \to \operatorname{Ker}\Sigma\,.$$

We will relate the groups $B_n(x)$ to more familiar objects.

Fix $x \in X^{n-1}$ and set $Y = V(x)$. Then Y is a geometrically reduced algebraic k-scheme [3, 4.6.4]. Further, the map $\overline{K}_1(x) \to Z^1(\overline{Y})$ factors through $\mathrm{Div}\overline{Y}$, the group of Cartier divisors on $\overline{Y}$. Consider

$$B'_n(x) = \mathrm{Ker}\left[H^2(k, \overline{K}_1(x)) \to H^2(k, \mathrm{Div}\overline{Y})\right] \subset B_n(x). \tag{9}$$

Let $\mathcal{R}^*_{\overline{Y}}$ denote the étale sheaf of invertible rational functions on $\overline{Y}$. Note that $\overline{K}_1(x) = H^0(\overline{Y}, \mathcal{R}^*_{\overline{Y}})$. Now, since $\overline{Y}$ is reduced, there exists an exact sequence of étale sheaves

$$0 \to \mathbb{G}_{m,\overline{Y}} \to \mathcal{R}^*_{\overline{Y}} \to \mathcal{D}iv_{\overline{Y}} \to 0,$$

where $\mathcal{D}iv_{\overline{Y}}$ is the sheaf of Cartier divisors on $\overline{Y}$ [3, 20.1.4 and 20.2.13]. This exact sequence gives rise to an exact sequence of étale cohomology groups

$$0 \to H^1_{\text{ét}}(\overline{Y}, \mathcal{D}iv_{\overline{Y}}) \to \mathrm{Br}'\,\overline{Y} \to H^2_{\text{ét}}(\overline{Y}, \mathcal{R}^*_{\overline{Y}}) \to H^2_{\text{ét}}(\overline{Y}, \mathcal{D}iv_{\overline{Y}}) \tag{10}$$

where $\mathrm{Br}'\,\overline{Y} = H^2_{\text{ét}}(\overline{Y}, \mathbb{G}_m)$ is the cohomological Brauer group of $\overline{Y}$ [2, II, p.73]. Similarly, there exists an exact sequence

$$0 \to H^1_{\text{ét}}(Y, \mathcal{D}iv_Y) \to \mathrm{Br}'\,Y \to H^2_{\text{ét}}(Y, \mathcal{R}^*_Y) \to H^2_{\text{ét}}(Y, \mathcal{D}iv_Y). \tag{11}$$

We will regard $H^1_{\text{ét}}(\overline{Y}, \mathcal{D}iv_{\overline{Y}})$ (resp. $H^1_{\text{ét}}(Y, \mathcal{D}iv_Y)$) as a subgroup of $\mathrm{Br}'\,\overline{Y}$ (resp. $\mathrm{Br}'\,Y$).

Now the exact sequence of terms of low degree

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to \mathrm{Ker}\,(E^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}$$

belonging to the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p\big(k, H^q_{\text{ét}}(\overline{Y}, \mathcal{R}^*_{\overline{Y}})\big) \implies H^{p+q}_{\text{ét}}(Y, \mathcal{R}^*_Y)$$

yields, using [2, II, Lemma 1.6, p.72], an exact sequence

$$0 \to H^2(k, \overline{K}_1(x)) \to H^2_{\text{ét}}(Y, \mathcal{R}^*_Y) \to H^2_{\text{ét}}(\overline{Y}, \mathcal{R}^*_{\overline{Y}}). \tag{12}$$

Similarly, the spectral sequence

$$H^p\big(k, H^q_{\text{ét}}(\overline{Y}, \mathcal{D}iv_{\overline{Y}})\big) \implies H^{p+q}_{\text{ét}}(Y, \mathcal{D}iv_Y)$$

yields a complex

$$\begin{array}{lll} 0 & \to & H^1(k, \mathrm{Div}\overline{Y}) \to H^1_{\text{ét}}(Y, \mathcal{D}iv_Y) \xrightarrow{\psi} H^1_{\text{ét}}(\overline{Y}, \mathcal{D}iv_{\overline{Y}})^{\Gamma} \\ & \xrightarrow{\varphi} & H^2(k, \mathrm{Div}\overline{Y}) \to H^2_{\text{ét}}(Y, \mathcal{D}iv_Y) \to H^2_{\text{ét}}(\overline{Y}, \mathcal{D}iv_{\overline{Y}}) \end{array} \tag{13}$$

which is exact except perhaps at $H^2_{\text{ét}}(Y, \mathcal{D}iv_Y)$. The map labeled ψ in (13) is induced by the canonical morphism $\overline{Y} \to Y$, while the map φ is the differential $d_2^{0,1}$ coming from the spectral sequence (see [6, II.4, pp.39-52]). Now we have a commutative diagram

(14)
$$\begin{tikzcd} 0 \arrow[r] & H^2(k,\overline{K}_1(x)) \arrow[r] \arrow[d] & H^2_{\text{ét}}(Y,\mathcal{R}^*_Y) \arrow[r] \arrow[d] & H^2_{\text{ét}}\big(\overline{Y},\mathcal{R}^*_{\overline{Y}}\big) \arrow[d] \\ 0 \arrow[r] & H^2\big(k,\text{Div}\,\overline{Y}\big)/\text{Im}\,\varphi \arrow[r] & H^2_{\text{ét}}(Y,\mathcal{D}iv_Y) \arrow[r] & H^2_{\text{ét}}\big(\overline{Y},\mathcal{D}iv_{\overline{Y}}\big). \end{tikzcd}$$

in which the top row is the exact sequence (12), the bottom row (which is only a complex) is derived from (13), and the middle and right-hand vertical maps are the maps in (11) and (10), respectively. Set

$$\widehat{\text{Br}'_1}\,Y=\text{Ker}\left[\text{Br}'\,Y/H^1_{\text{ét}}(Y,\mathcal{D}iv_Y)\to\text{Br}'\,\overline{Y}/H^1_{\text{ét}}\big(\overline{Y},\mathcal{D}iv_{\overline{Y}}\big)\right].$$

Then the above diagram yields a natural isomorphism

$$\widehat{\text{Br}'_1}\,Y=\text{Ker}\big[H^2(k,\overline{K}_1(x))\to H^2\big(k,\text{Div}\,\overline{Y}\big)/\text{Im}\,\varphi\big]\,. \tag{15}$$

(Note: only the exactness of the top row of (14) is needed to obtain the above isomorphism.) On the other hand, there exists an obvious exact sequence

$$0\to B'_n(x)\to\text{Ker}\big[H^2(k,\overline{K}_1(x))\to H^2\big(k,\text{Div}\,\overline{Y}\big)/\text{Im}\,\varphi\big]\to\text{Im}\,\varphi,$$

where $B'_n(x)$ is the group (9). Using (15) and the fact that $\text{Im}\,\varphi$ is naturally isomorphic to $\text{Coker}\,\psi$, where ψ is the map appearing in (13), we conclude that there exists a natural exact sequence

$$0\to B'_n(x)\to\widehat{\text{Br}'_1}\,Y\xrightarrow{h}\text{Coker}\,\psi\,. \tag{16}$$

The map labeled h in the above exact sequence can be briefly described as "$\varphi^{-1}\circ h^2(\text{div})\circ u^{-1}$", where $u\colon H^2(k,\overline{K}_1(x))\to H^2_{\text{ét}}(Y,\mathcal{R}^*_Y)$ is the map intervening in (14) and $h^2(\text{div})\colon H^2(k,\overline{K}_1(x))\to H^2\big(k,\text{Div}\,\overline{Y}\big)$ is induced by $\text{div}\colon\overline{K}_1(x)\to\text{Div}\,\overline{Y}$. Next, set

$$\text{Br}'_1\,Y=\text{Ker}\big[\text{Br}'\,Y\to\text{Br}'\,\overline{Y}\big]\,.$$

There exists a natural exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\text{ét}}(Y,\mathcal{D}iv_Y) \arrow[r] \arrow[d, "\psi"] & \text{Br}'_1 Y \arrow[r, two heads] \arrow[d] & \text{Br}'_1 Y/H^1_{\text{ét}}(Y,\mathcal{D}iv_Y) \arrow[d] \\ 0 \arrow[r] & H^1_{\text{ét}}\big(\overline{Y},\mathcal{D}iv_{\overline{Y}}\big)^{\Gamma} \arrow[r] & \big(\text{Br}'\,\overline{Y}\big)^{\Gamma} \arrow[r] & \big(\text{Br}'\,\overline{Y}/H^1_{\text{ét}}\big(\overline{Y},\mathcal{D}iv_{\overline{Y}}\big)\big)^{\Gamma}\,. \end{tikzcd}$$

An application of the snake lemma to the above diagram yields a natural exact sequence

$$0\to H^1(k,\text{Div}\,\overline{Y})\to\text{Br}'_1\,Y\to\widehat{\text{Br}'_1}\,Y\xrightarrow{\delta}\text{Coker}\,\psi\,. \tag{17}$$

Now using the explicit description of the map δ [8, Lemma 1.3.2, p.11] together with the description of the map $\varphi = d_2^{0,1}$ from [6, §II.4], it can be shown (with some work) that the maps h in (16) and δ in (17) *are the same.* Thus we obtain

Proposition 4.1. *There exists a canonical isomorphism*

$$B'_n(x) = \mathrm{Br}'_1 Y / H^1\big(k, \mathrm{Div}\overline{Y}\big).$$

Corollary 4.2. *Let $x \in X^{n-1}$ be such that $\overline{Y} = V(\overline{x})$ is* locally factorial *(this holds, for example, if $Y = V(x)$ is regular). Then there exists a canonical isomorphism*

$$B_n(x) = \mathrm{Br}'_1 Y.$$

Proof. The hypothesis implies that $\mathrm{Div}\overline{Y} = Z^1\big(\overline{Y}\big)$ [3, 21.6.9], so $B_n(x) = B'_n(x)$. On the other hand, since $Z^1\big(\overline{Y}\big)$ is a permutation Γ-module, $H^1\big(k, \mathrm{Div}\overline{Y}\big) = H^1\big(k, Z^1\big(\overline{Y}\big)\big) = 0$. The result is now immediate from the proposition. □

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Departamento de Matemáticas, Universidad de La Serena, La Serena, Chile

E-mail address: cgonzalez@userena.cl